

• Method of variation of Parameters

[This method is used for finding the complete solution of $y'' + Py' + Qy = R$. when the C.F. is known.]

Solution:— Let $y = Au + Bv$ be the C.F. of the 2nd order linear diff. eqn.

$$y'' + Py' + Qy = R \dots \dots \dots (1)$$

where A and B are constants and u and v are functions of x. Then obviously u and v are the solutions of

$$y'' + Py' + Qy = 0 \dots \dots \dots (2)$$

Then,
$$\left. \begin{aligned} u'' + Pu' + Qu = 0 \\ v'' + Pv' + Qv = 0 \end{aligned} \right\} \text{ and } \dots \dots \dots (3)$$

When $R \neq 0$, $Au + Bv$ do not represent the complete solution of (1).

Now, let us assume that

$$y = Au + Bv \dots \dots \dots (4)$$

is the complete solution of (1), where A and B are not constants but functions of x.

Differentiating (4), we get

$$y' = Au' + A'u + Bv' + B'v$$

$$\Rightarrow y' = Au' + Bv' + (A'u + B'v)$$

Now, we choose that $A'u + B'v = 0 \dots \dots \dots (5)$

such that $y' = Au' + Bv' \dots \dots \dots (6)$

Differentiating (6), we get

$$y'' = Au'' + A'u' + Bv'' + B'v' \dots \dots \dots (7)$$

Now putting the values of y, y' and y'' from (4), (6) and (7) in (1), we get

$$(Au'' + A'u' + Bv'' + B'v') + P(Au' + Bv') + Q(Au + Bv) = R$$

$$\Rightarrow A(u'' + Pu' + Qu) + B(v'' + Pv' + Qv) + A'u' + B'v' = R$$

$$\Rightarrow A \cdot 0 + B \cdot 0 + A'u' + B'v' = R$$

$$\therefore A'u' + B'v' = R \dots \dots \dots (8)$$

Here, $A'u + B'v + 0 = 0 \dots \dots \dots (5)$ and

$$A'u' + B'v' - R = 0 \dots \dots \dots (8)$$

Solving (5) and (8) by Cross-Multiplication, we have

$$\frac{A'}{-QR} = \frac{B'}{uR} = \frac{1}{uv' - vu'}$$

$$\therefore \frac{dA}{dx} = A' = -\frac{QR}{uv' - vu'} \text{ and}$$

$$\frac{dB}{dx} = B' = \frac{uR}{uv' - vu'}$$

$$\Rightarrow dA = -\frac{QR}{uv' - vu'} dx, \quad dB = \frac{uR}{uv' - vu'} dx$$

Integrating, we get

A = f(x) + C1 and B = g(x) + C2 (say)

Putting these values of A and B in (4), we get

y = C1u + C2u + u f(x) + v g(x)

where C1 and C2 are arbitrary constants. . . which is the required solution

Q.2). Apply the method of variation of parameters to solve:

d^2y/dx^2 + n^2y = sec nx

Solution :- Here a.e. is

(D^2 + n^2) = 0 => D^2 = -n^2 = i^2 n^2

∴ D = ± in

Then C.F. = A cos nx + B sin nx

where A and B are constants. (1)

Let y = A cos nx + B sin nx be the complete solution of the given equation where A and B are functions of x. such that

dy/dx = -nA sin nx + cos nx . dA/dx + nB cos nx + sin nx . dB/dx = -nA sin(nx) + nB cos(nx) + dA/dx . cos nx + dB/dx . sin(nx)

Now, choosing A and B such that

dA/dx . cos nx + dB/dx sin nx = 0 (2)

We have

dy/dx = -An sin nx + Bn cos nx

$$\Rightarrow \frac{d^2 y}{dx^2} = -An^2 \cos nx - Bn^2 \sin nx - n \frac{dA}{dx} \sin nx + n \frac{dB}{dx} \cos nx. \quad (28)$$

Substituting these values in the given equation, we have

$$-n^2(A \cos nx + B \sin nx) - n \frac{dA}{dx} \sin nx + n \frac{dB}{dx} \cos nx + n^2 y = \sec nx$$

$$\Rightarrow -n^2 y - n \frac{dA}{dx} \sin nx + n \frac{dB}{dx} \cos nx + n^2 y = \sec nx$$

$$\Rightarrow -n \frac{dA}{dx} \sin nx + n \frac{dB}{dx} \cos nx = \sec nx \quad \text{[from (1)]} \quad (3)$$

Now multiplying (2) by $n \sin nx$ and (3) by $\cos nx$ and adding, we have

$$n \frac{dB}{dx} = 1 \Rightarrow \frac{dB}{dx} = \frac{1}{n}, \text{ Integrating}$$

$$B = \frac{x}{n} + C_2$$

Again, multiplying (2) by $n \cos nx$ and (3) by $\sin nx$ and subtracting, we get

$$n \frac{dA}{dx} = -\tan nx \Rightarrow \frac{dA}{dx} = -\frac{1}{n} (\tan nx)$$

Integrating, we get

$$A = \frac{1}{n^2} \log \cos nx + C_1$$

Substituting the values of A and B in (1), we get the complete solution

$$y = C_1 \cos nx + C_2 \sin nx + \frac{1}{n^2} \cos nx \cdot \log \cos nx + \frac{x}{n} \sin nx.$$

By
Dr. Birkrama Singh
Associate Professor
Dept. of Maths
Sri Shankar College, Sasaram.
06.06.2020.